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POINT INTERVAL ESTIMATION, FROM ONE-ORDER STATISTIC, OF THE LOCATION PARAMETER OF AN EXTREME-VALUE DISTRIBUTION WITH KNOWN SCALE PARAMETER AND OF THE SCALE PARAMETER OF A WEIBULL DISTRIBUTION WITH KNOWN SHAPE PARAMETER

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Point and Interval Estimation, From One-Order Statistic, of the Location Parameter of an Extreme-Value Distribution with Known Scale Parameter and of the Scale Parameter of a Weibull Distribution with Known Shape Parameter

ALBERT H. MOORE AND H. LEON HARTER

Abstract—This paper derives a one-order statistic estimator $\tilde{u}_{mn}|b$ for the location parameter of the (first) extreme-value distribution of smallest values with cumulative distribution function $F(x;u,b) = 1 - \exp\{-\exp[(x-u)/b]\}$ using the minimum-variance unbiased one-order statistic estimator for the scale parameter of an exponential distribution, as was done in an earlier paper for the scale parameter of a Weibull distribution. It is shown that exact confidence bounds, based on one-order statistic, can be easily derived for the location parameter of the extreme-value distribution and for the scale parameter of the Weibull distribution, using exact confidence bounds for the scale parameter of the exponential distribution. The estimator for u is shown to be $b \ln c_{mn} + x_{mn}$ where x_{mn} is the mth order statistic from an ordered sample of size n from the extreme-value distribution with scale parameter b and c_{mn} is the coefficient for a one-order statistic estimator of the scale parameter of an exponential distribution. Values of the factor c_{mn}, which have previously been tabulated for $n = 1(1)20$, are given for $n = 21(1)40$. The ratios of the mean-square-errors of the maximum-likelihood estimators based on m order statistics to those of the one-order statistic estimators for the location parameter of the extreme-value distribution and the scale parameter of the Weibull distribution are investigated by Monte Carlo methods. The use of the table and related tables is discussed and illustrated by numerical examples.

I. Introduction

In a previous paper, Harter and Moore [1] have derived a maximum-likelihood and an unbiased estimator $\hat{u}|b$ and $\bar{u}|b$ of the location parameter of the extreme-value distribution with known scale parameter, based on the first m out of n ordered observations. However, in many practical applications an inefficient estimator may be chosen for its inherent simplicity. Harter [2] found the minimum-variance unbiased one-order statistic estimator for the scale parameter σ of the exponential distribution, from a sample of size n. Moore and Harter [3] tabulated the coefficient c_{mn} of the minimum-variance unbiased one-order statistic estimator for the scale parameter of the exponential distribution from a censored sample of size m from a life test of n items $[n = 1(1)20]$ and showed how it could be used to obtain a one-order statistic estimator for the scale parameter of Weibull populations with known shape parameter. By use of the coefficient of the mth order statistic, computed for estimation of the scale parameter of the exponential distribution, it is shown in Section II that a consistent one-order statistic estimator for the scale parameter of the extreme-value distribution with known scale parameter can be obtained. The values of c_{kn} are given in Table I, along with the relative efficiencies of the one-order statistic estimators of the scale parameter σ of an exponential distribution as compared with the unbiased m-order statistic estimators, for $n = 21(1)40$, $m = 1(1)n$, and $k = \min(m,r)$, where the rth order statistic is optimal for the complete sample. In Sections III and IV it is shown that exact confidence bounds, based on one-order statistic, can be derived for the location parameter of the extreme-value distribution and the scale parameter of the Weibull distribution using the coefficients of the exact confidence bounds, found by Harter [4], for the scale parameter of an exponential distribution. In Section V a Monte Carlo comparison of the relative merits of the one-order statistic estimators and the maximum-likelihood estimators is given. In Section VI, the use of Table I and related tables is discussed and illustrated by numerical examples.

II. Mathematical Formulation

If Y has an exponential distribution with scale parameter σ and location parameter zero, then $X = b \ln Y$ has the (first) extreme-value distribution with $u = b \ln \sigma$ as location parameter and b as scale parameter. A one-order statistic estimator for the scale parameter of the exponential distribution is given by

$$\tilde{\sigma} = c_{mn} y_{mn} \tag{1}$$

where

$$c_{mn} = 1\Big/\left[\sum_{i=1}^{m} 1/(n-i+1)\right] \tag{2}$$

and y_{mn} is the mth order statistic from an ordered sample of size n from the exponential distribution. Therefore, an

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TABLE 1

Coefficient of Optimum (Kth) Order Statistic in Unbiased Estimator, Based on One Order Statistic, of Parameter σ of One-Parameter Exponential Population, Given First M-order Statistics of Sample of Size N, and its Efficiency Relative to Best M-Order-Statistic Estimator

N	K	M	C(K, N)	EFF.
21	1	1	21.00000	100.00
21	2	2	10.24390	99.94
21	3	3	6.65555	99.83
21	4	4	4.85894	99.67
21	5	5	3.77887	99.44
21	6	6	3.05689	99.14
21	7	7	2.53938	98.75
21	8	8	2.14950	98.25
21	9	9	1.84451	97.62
21	10	10	1.59877	96.84
21	11	11	1.39589	95.88
21	12	12	1.22490	94.68
21	13	13	1.07817	93.19
21	14	14	0.95012	91.33
21	15	15	0.83657	88.99
21	16	16	0.73420	86.00
21	17	17	0.64019	82.10
21	17	18	0.64019	77.54
21	17	19	0.64019	73.46
21	17	20	0.64019	69.78
21	17	21	0.64019	66.46
22	1	1	22.00000	100.00
22	2	2	10.74419	99.95
22	3	3	6.98041	99.85
22	4	4	5.10073	99.70
22	5	5	3.97003	99.50
22	6	6	3.22493	99.23
22	7	7	2.68396	98.88
22	8	8	2.27660	98.44
22	9	9	1.95818	97.80
22	10	10	1.70183	97.21
22	11	11	1.49046	96.37
22	12	12	1.31260	95.34
22	13	13	1.16030	94.08
22	14	14	1.02780	92.53
22	15	15	0.91078	90.61
22	16	16	0.80592	88.19
22	17	17	0.71049	85.13
22	18	18	0.62209	81.16
22	18	19	0.62209	76.89
22	18	20	0.62209	73.04
22	18	21	0.62209	69.56
22	18	22	0.62209	66.4[illegible]
23	1	1	23.00000	100.00
23	2	2	11.24444	99.95
23	3	3	7.32323	99.86
23	4	4	5.36044	99.73
23	5	5	4.18089	99.54
23	6	6	3.39283	99.30
23	7	7	2.82835	98.90
23	8	8	2.40349	98.50
23	9	9	2.07155	98.11
23	10	10	1.80454	97.51
23	11	11	1.58458	96.78
23	12	12	1.39975	95.89
23	13	13	1.24174	94.81
23	14	14	1.10458	93.49
23	15	15	0.98383	91.88
23	16	16	0.87609	89.89
23	17	17	0.77864	87.41
23	18	18	0.68920	84.28
23	19	19	0.60571	80.24
23	19	20	0.60571	76.23
23	19	21	0.60571	72.60
23	19	22	0.60571	69.30
23	19	23	0.60571	66.29
24	1	1	24.00000	100.00
24	2	2	11.74468	99.95
24	3	3	7.65700	99.87
24	4	4	5.61109	99.75
24	5	5	4.38177	99.58
24	6	6	3.56062	99.36
24	7	7	2.97260	99.08
24	8	8	2.53018	98.73
24	9	9	2.18470	98.30
24	10	10	1.90606	97.77
24	11	11	1.67835	97.12
24	12	12	1.48644	96.34
24	13	13	1.32261	95.41
24	14	14	1.18065	94.27
24	15	15	1.05598	92.90
24	16	16	0.94500	91.23
24	17	17	0.84524	89.18
24	18	18	0.75417	86.64
24	19	19	0.66996	83.45
24	20	20	0.59080	79.36
24	20	21	0.59080	75.58
24	20	22	0.59080	72.14
24	20	23	0.59080	69.01
24	20	24	0.59080	66.13
25	1	1	25.00000	100.00
25	2	2	12.24490	99.96
25	3	3	7.99074	99.88
25	4	4	5.86168	99.77
25	5	5	4.58256	99.62
25	6	6	3.72830	99.42
25	7	7	3.11672	99.16
25	8	8	2.65671	98.85
25	9	9	2.29764	98.46
25	10	10	2.00012	97.98
25	11	11	1.77181	97.41
25	12	12	1.57276	96.73
25	13	13	1.40302	95.90
25	14	14	1.25615	94.92
25	15	15	1.12741	93.74
25	16	16	1.01318	92.31
25	17	17	0.91066	90.59
25	18	18	0.81759	88.48
25	19	19	0.73209	85.89
25	20	20	0.65248	82.64
25	20	21	0.65248	78.71
25	20	22	0.65248	75.13
25	20	23	0.65248	71.86
25	20	24	0.65248	68.87
25	20	25	0.65248	66.12
26	1	1	26.00000	100.00
26	2	2	12.74510	99.96
26	3	3	8.32444	99.89
26	4	4	6.11223	99.79
26	5	5	4.78329	99.65
26	6	6	3.89590	99.47
26	7	7	3.26073	99.24
26	8	8	2.78310	98.95
26	9	9	2.41041	98.60
26	10	10	2.11108	98.17
26	11	11	1.86501	97.66
26	12	12	1.65877	97.05
26	13	13	1.48305	96.32
26	14	14	1.33119	95.46
26	15	15	1.19826	94.43
26	16	16	1.08055	93.20
26	17	17	0.97518	91.73
26	18	18	0.87985	89.95
26	19	19	0.79267	87.80
26	20	20	0.71204	85.15
26	21	21	0.63650	81.86
26	21	22	0.63650	78.14
26	21	23	0.63650	74.74
26	21	24	0.63650	71.62
26	21	25	0.63650	68.76
26	21	26	0.63650	66.11
27	1	1	27.00000	100.00
27	2	2	13.24528	99.96
27	3	3	8.65812	99.90
27	4	4	6.36273	99.81
27	5	5	4.98396	99.68
27	6	6	4.06342	99.51
27	7	7	3.40464	99.30
27	8	8	2.90937	99.04
27	9	9	2.52303	98.72

TABLE I (Cont.)

N	K	M	C(K, N)	EFF.
27	10	10	2.21286	98.33
27	11	11	1.95799	97.87
27	12	12	1.74451	97.33
27	13	13	1.56276	96.68
27	14	14	1.40583	95.92
27	15	15	1.26864	95.01
27	16	16	1.14734	93.94
27	17	17	1.03897	92.66
27	18	18	0.94119	91.15
27	19	19	0.85208	89.33
27	20	20	0.77006	87.13
27	21	21	0.69374	84.43
27	22	22	0.62184	81.00
27	22	23	0.62184	77.56
27	22	24	0.62184	74.33
27	22	25	0.62184	71.36
27	22	26	0.62184	68.61
27	22	27	0.62184	66.07
28	1	1	28.00000	100.00
28	2	2	13.74545	99.97
28	3	3	8.99177	99.91
28	4	4	6.61319	99.82
28	5	5	5.18458	99.70
28	6	6	4.23087	99.55
28	7	7	3.54846	99.35
28	8	8	3.03553	99.12
28	9	9	2.63552	98.82
28	10	10	2.31448	98.48
28	11	11	2.05078	98.06
28	12	12	1.83002	97.57
28	13	13	1.64219	96.99
28	14	14	1.48015	96.31
28	15	15	1.33862	95.50
28	16	16	1.21365	94.56
28	17	17	1.10218	93.45
28	18	18	1.00180	92.[illegible]3
28	19	19	0.91058	90.57
28	20	20	0.82692	88.71
28	21	21	0.74945	86.47
28	22	22	0.67697	83.73
28	23	23	0.60833	80.34
28	23	24	0.60833	76.99
28	23	25	0.60833	73.91
28	23	26	0.60833	71.07
28	23	27	0.60833	68.44
28	23	28	0.60833	65.99
29	1	1	29.00000	100.00
29	2	2	14.24561	99.97
29	3	3	9.32539	99.91
29	4	4	6.86362	99.83
29	5	5	5.38516	99.72
29	6	6	4.39827	99.58
29	7	7	3.69221	99.40
29	8	8	3.16160	99.18
29	9	9	2.74790	98.92
29	10	10	2.41596	98.60
29	11	11	2.14341	98.22
29	12	12	1.91534	97.78
29	13	13	1.72139	97.25
29	14	14	1.55418	96.64
29	15	15	1.40827	95.93
29	16	16	1.27958	95.09
29	17	17	1.16490	94.11
29	18	18	1.06182	92.96
29	19	19	0.96835	91.61
29	20	20	0.88286	90.01
29	21	21	0.80399	88.10
29	22	22	0.73057	85.81
29	23	23	0.66153	83.04
29	24	24	0.59583	79.61
29	24	25	0.59583	76.42
29	24	26	0.59583	73.48
29	24	27	0.59583	70.76
29	24	28	0.59583	68.24
29	24	29	0.59583	65.88
30	1	1	30.00000	100.00
30	2	2	14.74576	99.97
30	3	3	9.65900	99.92
30	4	4	7.11402	99.85
30	5	5	5.58569	99.74
30	6	6	4.56561	99.61
30	7	7	3.83589	99.45
30	8	8	3.28759	99.25
30	9	9	2.86018	99.00
30	10	10	2.51732	98.71
30	11	11	2.23590	98.36
30	12	12	2.00048	97.96
30	13	13	1.80039	97.49
30	14	14	1.62798	96.93
30	15	15	1.47763	96.29
30	16	16	1.34513	95.54
30	17	17	1.22721	94.67
30	18	18	1.12136	93.66
30	19	19	1.02553	92.47
30	20	20	0.93807	91.08
30	21	21	0.85762	89.45
30	22	22	0.78301	87.50
30	23	23	0.71320	85.18
30	24	24	0.64725	82.36
30	24	25	0.64725	79.06
30	24	26	0.64725	76.02
30	24	27	0.64725	73.21
30	24	28	0.64725	70.59
30	24	29	0.64725	68.16
30	24	30	0.64725	65.89
31	1	1	31.00000	100.00
31	2	2	15.24590	99.97
31	3	3	9.99259	99.93
31	4	4	7.36440	99.86
31	5	5	5.78618	99.76
31	6	6	4.73290	99.64
31	7	7	3.97951	99.49
31	8	8	3.41351	99.30
31	9	9	2.97237	99.07
31	10	10	2.61858	98.81
31	11	11	2.32826	98.49
31	12	12	2.08548	98.12
31	13	13	1.87921	97.69
31	14	14	1.70157	97.19
31	15	15	1.54675	96.61
31	16	16	1.41041	95.93
31	17	17	1.28919	95.16
31	18	18	1.18048	94.25
31	19	19	1.08221	93.21
31	20	20	0.99269	91.99
31	21	21	0.91052	90.56
31	22	22	0.83453	88.89
31	23	23	0.76372	86.91
31	24	24	0.69716	84.55
31	25	25	0.63402	81.70
31	25	26	0.63402	78.55
31	25	27	0.63402	75.65
31	25	28	0.63402	72.94
31	25	29	0.63402	70.43
31	25	30	0.63402	68.08
31	25	31	0.63402	65.88
32	1	1	32.00000	100.00
32	2	2	15.74603	99.97
32	3	3	10.32616	99.93
32	4	4	7.61475	99.87
32	5	5	5.98665	99.78
32	6	6	4.90015	99.66
32	7	7	4.12308	99.52
32	8	8	3.53936	99.35
32	9	9	3.08448	99.14
32	10	10	2.71974	98.89
32	11	11	2.42051	98.60
32	12	12	2.17035	98.26
32	13	13	1.95788	97.87
32	14	14	1.77498	97.41
32	15	15	1.61566	96.88
32	16	16	1.47544	96.28
32	17	17	1.35087	95.58
32	18	18	1.23926	94.77

TABLE I (Cont.)

N	K	M	C(K, N)	EFF.
32	19	19	1.13848	93.84
32	20	20	1.04681	92.76
32	21	21	0.96282	91.51
32	22	22	0.88533	90.05
32	23	23	0.81332	88.34
32	24	24	0.74501	86.33
32	25	25	0.68230	83.95
32	26	26	0.62170	81.05
32	26	27	0.62170	78.05
32	26	28	0.62170	75.26
32	26	29	0.62170	72.66
32	26	30	0.62170	70.24
32	26	31	0.62170	67.98
32	26	32	0.62170	65.85
33	1	1	33.00000	100.00
33	2	2	16.24615	99.98
33	3	3	10.65972	99.93
33	4	4	7.86597	99.87
33	5	5	6.18708	99.79
33	6	6	5.06736	99.69
33	7	7	4.26660	99.55
33	8	8	3.66515	99.39
33	9	9	3.19652	99.20
33	10	10	2.82082	98.97
33	11	11	2.51266	98.70
33	12	12	2.25510	98.39
33	13	13	2.03642	98.03
33	14	14	1.84823	97.61
33	15	15	1.68438	97.13
33	16	16	1.54025	96.58
33	17	17	1.41229	95.94
33	18	18	1.29774	95.22
33	19	19	1.19441	94.38
33	20	20	1.10052	93.42
33	21	21	1.01462	92.31
33	22	22	0.93552	91.03
33	23	23	0.86220	89.54
33	24	24	0.79376	87.80
33	25	25	0.72943	85.76
33	26	26	0.66848	83.33
33	27	27	0.61020	80.41
33	27	28	0.61020	77.54
33	27	29	0.61020	74.87
33	27	30	0.61020	72.37
33	27	31	0.61020	70.04
33	27	32	0.61020	67.85
33	27	33	0.61020	65.79
34	1	1	34.00000	100.00
34	2	2	16.74627	99.98
34	3	3	10.99326	99.94
34	4	4	8.11538	99.88
34	5	5	6.38748	99.80
34	6	6	5.23453	99.71
34	7	7	4.41008	99.58
34	8	8	3.79069	99.43
34	9	9	3.30850	99.25
34	10	10	2.92183	99.04
34	11	11	2.60472	98.79
34	12	12	2.33975	98.50
34	13	13	2.11483	98.17
34	14	14	1.92134	97.79
34	15	15	1.75294	97.35
34	16	16	1.60487	96.84
34	17	17	1.47350	96.27
34	18	18	1.35597	95.61
34	19	19	1.25003	94.85
34	20	20	1.15387	93.99
34	21	21	1.06601	93.00
34	22	22	0.98522	91.87
34	23	23	0.91047	90.56
34	24	24	0.84087	89.04
34	25	25	0.77565	87.27
34	26	26	0.71411	85.20
34	27	27	0.65559	82.74
34	28	28	0.59944	79.79
34	28	29	0.59944	77.04
34	28	30	0.59944	74.47
34	28	31	0.59944	72.07
34	28	32	0.59944	69.82
34	28	33	0.59944	67.70
34	28	34	0.59944	65.71
35	1	1	35.00000	100.00
35	2	2	17.24638	99.98
35	3	3	11.32680	99.94
35	4	4	8.36566	99.89
35	5	5	6.58786	99.82
35	6	6	5.40168	99.72
35	7	7	4.55352	99.61
35	8	8	3.91658	99.47
35	9	9	3.42042	99.30
35	10	10	3.02276	99.10
35	11	11	2.69670	98.87
35	12	12	2.42430	98.60
35	13	13	2.19314	98.30
35	14	14	1.99433	97.94
35	15	15	1.82136	97.54
35	16	16	1.66933	97.08
35	17	17	1.53451	96.55
35	18	18	1.41397	95.95
35	19	19	1.30530	95.27
35	20	20	1.20692	94.49
35	21	21	1.11705	93.60
35	22	22	1.03450	92.59
35	23	23	0.95825	91.43
35	24	24	0.88739	90.09
35	25	25	0.82114	88.54
35	26	26	0.75883	86.75
35	27	27	0.69983	84.64
35	28	28	0.64353	82.16
35	28	29	0.64353	79.32
35	28	30	0.64353	76.68
35	28	31	0.64353	74.21
35	28	32	0.64353	71.89
35	28	33	0.64353	69.71
35	28	34	0.64353	67.66
35	28	35	0.64353	65.72
36	1	1	36.00000	100.00
36	2	2	17.74648	99.98
36	3	3	11.66032	99.95
36	4	4	8.61594	99.89
36	5	5	6.78822	99.83
36	6	6	5.56879	99.74
36	7	7	4.69692	99.63
36	8	8	4.04223	99.50
36	9	9	3.53229	99.34
36	10	10	3.12364	99.16
36	11	11	2.78861	98.94
36	12	12	2.50877	98.70
36	13	13	2.27135	98.41
36	14	14	2.06720	98.08
36	15	15	1.88964	97.71
36	16	16	1.73364	97.29
36	17	17	1.59536	96.80
36	18	18	1.47178	96.25
36	19	19	1.36053	95.63
36	20	20	1.25972	94.93
36	21	21	1.16777	94.13
36	22	22	1.08343	93.21
36	23	23	1.00561	92.18
36	24	24	0.93340	90.99
36	25	25	0.86604	89.62
36	26	26	0.80283	88.05
36	27	27	0.74317	86.23
36	28	28	0.68648	84.10
36	29	29	0.63223	81.58
36	29	30	0.63223	78.86
36	29	31	0.63223	76.32
36	29	32	0.63223	73.94
36	29	33	0.63223	71.70
36	29	34	0.63223	69.59
36	29	35	0.63223	67.60
36	29	36	0.63223	65.72
37	1	1	37.00000	100.00
37	2	2	18.24658	99.98

TABLE I (Cont.)

N	K	M	C(K, N)	EFF.
37	3	3	11.99383	99.95
37	4	4	8.86619	99.90
37	5	5	6.99856	99.84
37	6	6	5.73588	99.75
37	7	7	4.84029	99.65
37	8	8	4.16784	99.53
37	9	9	3.64411	99.38
37	10	10	3.22446	99.21
37	11	11	2.88046	99.01
37	12	12	2.59317	98.78
37	13	13	2.34947	98.51
37	14	14	2.13968	98.21
37	15	15	1.95782	97.86
37	16	16	1.79783	97.47
37	17	17	1.65605	97.03
37	18	18	1.52941	96.53
37	19	19	1.41547	95.96
37	20	20	1.31228	95.31
37	21	21	1.21824	94.59
37	22	22	1.13204	93.76
37	23	23	1.05261	92.83
37	24	24	0.97900	91.76
37	25	25	0.91044	90.55
37	26	26	0.84623	89.16
37	27	27	0.78578	87.57
37	28	28	0.72853	85.72
37	29	29	0.67398	83.56
37	30	30	0.62161	81.02
37	30	31	0.62161	78.41
37	30	32	0.62161	75.96
37	30	33	0.62161	73.66
37	30	34	0.62161	71.49
37	30	35	0.62161	69.45
37	30	36	0.62161	67.52
37	30	37	0.62161	65.69
38	1	1	38.00000	100.00
38	2	2	18.74667	99.98
38	3	3	12.32733	99.95
38	4	4	9.11643	99.91
38	5	5	7.18888	99.85
38	6	6	5.90295	99.77
38	7	7	4.98363	99.67
38	8	8	4.29341	99.56
38	9	9	3.75589	99.42
38	10	10	3.32523	99.26
38	11	11	2.97225	99.07
38	12	12	2.67750	98.85
38	13	13	2.42752	98.61
38	14	14	2.21266	98.32
38	15	15	2.02589	98.00
38	16	16	1.86189	97.64
38	17	17	1.71661	97.23
38	18	18	1.58689	96.77
38	19	19	1.47024	96.25
38	20	20	1.36464	95.66
38	21	21	1.26847	94.99
38	22	22	1.18040	94.24
38	23	23	1.09930	93.40
38	24	24	1.02423	92.44
38	25	25	0.95441	91.36
38	26	26	0.88913	90.12
38	27	27	0.82780	88.71
38	28	28	0.76986	87.09
38	29	29	0.71483	85.21
38	30	30	0.66223	83.03
38	31	31	0.61160	80.47
38	31	32	0.61160	77.96
38	31	33	0.61160	75.60
38	31	34	0.61160	73.37
38	31	35	0.61160	71.28
38	31	36	0.61160	69.30
38	31	37	0.61160	67.42
38	31	38	0.61160	65.65
39	1	1	39.00000	100.00
39	2	2	19.24675	99.98
39	3	3	12.66082	99.95
39	4	4	9.36666	99.91
39	5	5	7.38918	99.85
39	6	6	6.06909	99.78
39	7	7	5.12695	99.69
39	8	8	4.41896	99.58
39	9	9	3.86764	99.45
39	10	10	3.42506	99.30
39	11	11	3.08399	99.12
39	12	12	2.76177	98.92
39	13	13	2.50549	98.69
39	14	14	2.28527	98.43
39	15	15	2.09387	98.13
39	16	16	1.92565	97.79
39	17	17	1.77705	97.41
39	18	18	1.64424	96.99
39	19	19	1.52485	96.59
39	20	20	1.41683	95.96
39	21	21	1.31856	95.36
39	22	22	1.22852	94.67
39	23	23	1.14572	93.90
39	24	24	1.06916	93.04
39	25	25	0.99802	92.08
39	26	26	0.93161	90.95
39	27	27	0.86931	89.69
39	28	28	0.81059	88.26
39	29	29	0.75496	86.61
39	30	30	0.70196	84.72
39	31	31	0.65117	82.51
39	31	32	0.65117	79.94
39	31	33	0.65117	77.51
39	31	34	0.65117	75.23
39	31	35	0.65117	73.08
39	31	36	0.65117	71.05
39	31	37	0.65117	69.13
39	31	38	0.65117	67.31
39	31	39	0.65117	65.59
40	1	1	40.00000	100.00
40	2	2	19.74684	99.98
40	3	3	12.99430	99.96
40	4	4	9.61688	99.92
40	5	5	7.58946	99.86
40	6	6	6.23702	99.79
40	7	7	5.27024	99.71
40	8	8	4.54447	99.60
40	9	9	3.97934	99.48
40	10	10	3.52664	99.34
40	11	11	3.15568	99.17
40	12	12	2.84599	98.99
40	13	13	2.58340	98.77
40	14	14	2.35780	98.52
40	15	15	2.16177	98.24
40	16	16	1.98971	97.93
40	17	17	1.83739	97.58
40	18	18	1.70146	97.18
40	19	19	1.57932	96.74
40	20	20	1.46885	96.24
40	21	21	1.36836	95.68
40	22	22	1.27643	95.05
40	23	23	1.19191	94.35
40	24	24	1.11382	93.56
40	25	25	1.04133	92.68
40	26	26	0.97373	91.68
40	27	27	0.91041	90.55
40	28	28	0.85082	89.27
40	29	29	0.79449	87.81
40	30	30	0.74097	86.14
40	31	31	0.68986	84.23
40	32	32	0.64074	82.00
40	32	33	0.64074	79.52
40	32	34	0.64074	77.18
40	32	35	0.64074	74.97
40	32	36	0.64074	72.89
40	32	37	0.64074	70.92
40	32	38	0.64074	69.06
40	32	39	0.64074	67.28
40	32	40	0.64074	65.60

estimator for u is given by

$$\begin{aligned}\hat{u} &= b \ln \hat{\sigma} \\ &= b \ln (c_{mn} y_{mn}) \\ &= b \ln c_{mn} + b \ln y_{mn}\end{aligned}$$

and since $x = b \ln y$ we obtain

$$\hat{u} = b \ln c_{mn} + x_{mn} \tag{3}$$

where c_{mn} is given by (2) and x_{mn} is the mth order statistic from an ordered sample of size n from the extreme-value distribution. Now $\hat{u}$ is a consistent estimator of the location parameter u of the extreme-value distribution, since $u = b \ln \sigma$ and $\hat{\sigma}$ is a consistent estimator for the scale parameter σ of the exponential population.

Similarly, as was shown in an earlier paper [3], if Y has an exponential distribution with scale parameter σ then $T = Y^{1/K}$ has a Weibull distribution with shape parameter K, scale parameter $\theta = \sigma^{1/K}$, and

$$\hat{\theta} = c_{mn}^{1/K} t_{mn} \tag{4}$$

is a consistent estimator for θ and t_{mn} is the mth order statistic from a Weibull distribution with shape parameter K.

The coefficient c_{mn} has been tabulated for $n = 2(1)20$ by Moore and Harter [3]. The values of the coefficient are given for $n = 21(1)40$ in Table I, in which it is called c_{kn}, where $k = \min(m,r)$, the kth order statistic being optimal for estimation from the first m order statistics of a sample of size n and the rth order statistic being optimal for estimation from the complete sample.

III. Exact Confidence Bounds for the Location Parameter of the Extreme-Value Distribution

Harter [4] has obtained exact upper and lower bounds and central confidence intervals for the scale parameter of the one-parameter exponential distribution for a wide range of confidence levels based on the mth order statistic y_{mn} of a sample of size n. The coefficients $B_{um}\ y_{mn}$ have been tabulated for $n = 1(1)20(2)40$ for all m optimal. Let us introduce the notation

$$D_{\ell m} = B_{\ell m}/y_{mn} \text{ and } D_{um} = B_{um}/y_{mn}.$$

Now the exact confidence interval based on one-order statistic is given by

$$D_{\ell m} y_{mn} < \sigma < D_{um} y_{mn}. \tag{5}$$

Let $X = b \ln Y$ and we obtain

$$D_{\ell m} e^{x_{mn}/b} < \sigma < D_{um} e^{x_{mn}/b}$$

where x_{mn} is the mth order statistic in a sample of size n from the extreme-value distribution. Now take the natural logarithm of the terms of the inequality, multiply by b and we obtain

$$b \ln D_{\ell m} + x_{mn} < b \ln \sigma < b \ln D_{um} + x_{mn}.$$

But $u = b \ln \sigma$; therefore, by substitution, we obtain the following

$$b \ln D_{\ell m} + x_{mn} < u < b \ln D_{um} + x_{mn} \tag{6}$$

which gives an exact central confidence interval with the same level of confidence given by the tabulated values of $B_{\ell m}\ y_{mn} = D_{\ell m}$ and $B_{um}\ y_{mn} = D_{um}$. Therefore we have a simple method of computing exact central confidence intervals or upper and lower confidence bounds for the location parameter of the extreme-value distribution, with scale parameter b, based on one-order statistic.

IV. Exact Confidence Bounds for the Scale Parameter of the Weibull Distribution with Known Shape Parameter

If the random variable T has a Weibull distribution with shape parameter K then it is easily shown that $Y = T^K$ has an exponential distribution with $\theta = \sigma^{1/K}$. In inequality (5) replace y_{mn} by t_{mn}^K, the Kth power of the mth order statistic from a Weibull distribution with shape parameter K, and we obtain

$$D_{\ell m} t_{mn}^K < \sigma < D_{um} t_{mn}^K. \tag{7}$$

Take the Kth root of each member of (7) and obtain

$$D_{\ell m}^{1/K} t_{mn} < \sigma^{1/K} < D_{um}^{1/K} t_{mn}. \tag{8}$$

But $\theta = \sigma^{1/K}$ and therefore

$$D_{\ell m}^{1/K} t_{mn} < \theta < D_{um}^{1/K} t_{mn} \tag{9}$$

gives an exact central confidence interval for the scale parameter of Weibull distributions with known shape parameter K.

V. Monte Carlo Study of Ratios of Mean-Square-Errors

It seemed reasonable to the authors to conjecture that the ratios of the mean-square-errors of the m-order-statistic estimator and of the one-order statistic estimator for both the scale parameter of a two-parameter Weibull distribution with known shape parameter and the location parameter of an extreme-value distribution with known scale parameter are closely approximated by the relative efficiency of the one-order statistic estimator of the scale parameter of an one-parameter exponential distribution as compared with the m order statistic estimator, which has been tabulated by Moore and Harter [3] for $n = 1(1)20$ and in Table I of the present paper for $n = 21(1)40$. (It should be noted that one may speak of relative efficiency in the case of the exponential distribution, since the estimators are unbiased, but only of ratios of mean-square-errors in the cases of the Weibull and extreme-value distributions, for which the estimators are biased.) In order to check the validity of the conjecture, a Monte Carlo study of the ratios of mean-square-errors was performed. One thousand random samples each of size n [$n = 1(1)40$] from a one-parameter exponential distribution with scale

parameter 1 were generated in the IBM 7094 computer. These were transformed into samples from a 2-parameter Weibull distribution with shape parameter 2 and from an extreme-value distribution with scale parameter 1. From each sample, the one-order statistic estimate and the m order statistic estimate, based on the first m order statistics $[m = 1(1)n]$, of the scale parameters of the exponential and Weibull distributions and of the location parameter of the extreme-value distribution were computed. For each distribution and for each combination of m and n, the ratio of the mean-square-error of the m order statistic estimates to that of the one-order statistic estimates was calculated. Except for fluctuations due to random sampling, the ratio of mean-square-errors in the case of the exponential distribution should agree with the tabulated relative efficiencies, and it was found that the agreement was quite good. Moreover, it was found that the ratios of mean-square-errors in the cases of the Weibull and extreme-value distributions agreed with the tabulated relative efficiencies almost as well as did those for the exponential distribution, thus confirming the conjecture.

VI. Use of Table I and Related Tables, with Numerical Examples

Table I gives the coefficient of the optimum single-order statistic (the kth) in an unbiased estimator of the scale parameter of a one-parameter exponential distribution from the first m order statistics of a sample of size $n[n = 21(1)40]$, and the relative efficiency of the one-order statistic estimator as compared with the m order statistic estimator. It is a condensed extension of the similar table for $n = 1(1)20$ given by Moore and Harter [3], which also includes columns giving the variances of the two estimators. These columns have been omitted from Table I to save space, which can be done without loss of information, since the variance of the m order statistic estimator is simply $1/m$ and that of one-order statistic estimator can be found by dividing $1/m$ by the relative efficiency. These two tables can also be used to obtain consistent one-order statistic estimators of the scale parameter of a two-parameter Weibull distribution with known shape parameter and of the location parameter of an extreme-value distribution with known scale parameter, together with their approximate "efficiencies" (ratios of mean-square-errors) relative to the m order statistic estimators. Harter [4] has tabulated coefficients of optimum order statistics in exact upper and lower confidence bounds, based on one-order statistic, for the scale parameter of a one-parameter exponential distribution. These may also be used to obtain exact confidence bounds, based on one-order statistic, for the scale parameter of a two-parameter Weibull distribution with known shape parameter and for the location parameter of an extreme-value distribution with known scale parameter.

As an example of the previously mentioned uses of Table I and related tables, consider the following tabulation of data (observed failure times in hours) resulting from a simulated life test on forty components:

5	33	55	65	82	102	114	142
10	34	58	65	85	103	116	143
17	36	58	66	90	106	117	151
32	54	61	67	92	107	124	158
32	55	64	68	92	114	139	195

Suppose that the experimenter knows that these data have come from a two-parameter Weibull distribution with shape parameter $K = 2.0$, and that he wishes to find a point estimate and 80 percent lower and upper confidence bounds on the scale parameter θ. Harter and Moore [5] have previously done this for estimates based on the first m order statistics $[m = 8(8)40]$. From Table I of the present paper and from Table I of Harter [4], one finds that, for a one-parameter exponential distribution, the optimum-order statistic for obtaining a point estimator and the 80 percent lower and upper confidence bounds on the scale parameter is 32, with coefficients 0.64074, 0.553447, and 0.768717, respectively. Substituting these values in (4) and (9), one finds that the point estimate of the scale parameter of the Weibull distribution from which the above sample came is $\sqrt{0.64074(116)} = 92.9$, the 80 percent lower confidence bound is $\sqrt{0.553447(116)} = 86.3$ and the 80 percent upper confidence bound is $\sqrt{0.768717(116)} = 101.7$, as compared with results 93.7, 87.6, and 101.7 obtained from the first 32 order statistics, 93.3, 87.8, and 100.3 obtained from all 40 observations, and the true population parameter of 100.

Now consider the same data transformed to data from an extreme-value distribution with scale parameter $b = 0.5$ by taking natural logarithms.

1.609	3.497	4.007	4.174	4.407	4.625	4.736	4.956
2.303	3.526	4.060	4.174	4.443	4.635	4.754	4.963
2.833	3.584	4.060	4.190	4.500	4.663	4.762	5.017
3.466	3.989	4.111	4.205	4.522	4.673	4.820	5.063
3.466	4.007	4.159	4.220	4.522	4.736	4.934	5.273

Using the same tabular values, one finds by substituting in (3) and (6) that the point estimate of the location parameter of the extreme-value distribution is $0.5 \ln 0.64074 + 4.754 = 4.531$, the 80 percent lower confidence bound is $0.5 \ln 0.553447 + 4.754 = 4.458$, and the 80 percent upper confidence bound is $0.5 \ln 0.768717 + 4.754 = 4.623$, as compared with results [1] 4.541, 4.474, and 4.624 based on the first 32 order statistics, 4.537, 4.476, and 4.610 based on all 40 observations, and the true population parameter of 4.605 ($= \ln 100$).

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13. ABSTRACT

This paper derives a one-order statistic estimator $U_{mn}b$ for the location of the (first) extreme-value distribution of smallest values with cumulative distribution function $F(x;u,b) = 1 - \exp\{-\exp[(x-u)/b]\}$ using the minimum-variance unbiased one-order statistic estimator for the scale parameter of an exponential distribution, as was done in an earlier paper for the scale parameter of a Weibull distribution. It is shown that exact confidence bounds, based on one-order statistic, can be easily derived for the location parameter of the extreme-value distribution and for the scale parameter of the Weibull distribution, using exact confidence bounds for the scale parameter of the exponential distribution. The estimator for u is shown to be b in $c_{mn} + x_{mn}$ is the mth order statistic from an ordered sample of size n from the extreme -value distribution with scale parameter b and c_{mn} is the coefficient for a one-order statistic estimator of the scale parameter of an exponential distribution. Values of the factor c_{mn}, which have previously been tabulated for n = 1(1)20, are given for n = 21(1)40. The ratios of the mean-square-errors of the maximum-likelihood estimators based on m order statistics to those of the one-order statistic estimators for the location parameter of the extreme-value distribution and the scale parameter of the Weibull distribution are investigated by Monte Carlo methods. The use of the table and related tables is discussed and illustrated by numerical examples.

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